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Distributional Inequalities and Landau's Proof of the Weierstrass Theorem

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Distributional inequalities are shown to determine analytic, geometric, and convergence properties of the "Landau type polynomials" for a continuous function on a compact interval.

INTRODUCTION

After slightly altering the approximating polynomials in Landau's proof of the Weierstrass theorem we show that each of the redefined *Landau polynomials* $\tilde{f}_n$ for a continuous function f on $[a, b]$ has any nice property on $[a, b]$ that f has. This is done using distributional inequalities. Also, we prove that if the $m + 2$th distributional derivative of f is a measure, then $f \in C^m(a, b)$ and the sequence of mth derivatives $\{\tilde{f}_n^{(m)}\}$ converges uniformly to $f^{(m)}$ on compact subintervals of $[a, b]$.

1. NOTATION AND PRELIMINARIES

The space of test functions on the open interval (a, b) is denoted by $\mathscr{D}(a, b)$ and its dual space, $\mathscr{D}'(a, b)$, is the set of distributions in (a, b). For $f \in L^1_{loc}(a, b)$ (the space of locally Lebesgue integrable functions on (a, b)), T_f is its associated distribution in $\mathscr{D}'(a, b)$ and $\mathscr{D}^n T_f$ is the nth distributional derivative of f. For $T \in \mathscr{D}'(a, b)$, $T \geqslant 0$ in (a, b) means that T is positive semidefinite (when T is positive definite we write $T > 0$). For regular distributions T_f and T_g in $\mathscr{D}'(a, b)$, $\mathscr{D}^m T_f \geqslant T_g$ in (a, b) means $\mathscr{D}^m T_f - T_g \geqslant 0$ in (a, b).

$C^n(a, b)$ is the space of n-times continuously differentiable functions on (a, b) and $C[a, b]$ denotes the space of continuous functions on $[a, b]$.

A function f is said to be locally Lipschitzian on (a, b) if for each compact subinterval $[c, d]$ of (a, b) there exists a constant M (depending on $[c, d]$) such that $|f(x) - f(y)| \leqslant M|x - y|$ for all x and y in $[c, d]$.

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Let $Q_n(x)$ be $(1 - x^2)^n$ for $|x| \leq 1$ and 0 elsewhere. Set $1/c_n = \int_{-1}^{1} Q_n(x)\,dx$. Then $K_n = c_n Q_n$ forms a *Dirac sequence* $\{K_n\}$ and the functions K_n are called the *Landau kernels.*

For $f \in C[a, b]$, let L_f be the function determined by the line through the points $(a, f(a))$ and $(b, f(b))$, and let $\bar{f} = f - L_f$ on $[a, b]$ and vanish outside of $[a, b]$. Landau's proof of the Weierstrass theorem originally given in [5] (compare [2, p. 214]) essentially states: (1) If $f \in C[a, b]$, then $L_f + \bar{f} * \tilde{K}_n \to f$ uniformly on $[a, b]$; and (2) for each n, $L_f + \bar{f} * \tilde{K}_n$ is a polynomial of degree $\leq 2n$ on $[a, b]$, where $\tilde{K}_n(t) = (b - a)^{-1} K_n(t/(b - a))$ and $\bar{f} * \tilde{K}_n$ is the convolution

$$\int_{-\infty}^{+\infty} \bar{f}(t)\tilde{K}_n(x - t)\,dt = \int_a^b \bar{f}(t)\tilde{K}_n(x - t)\,dt.$$

Define $\hat{f}$ by

$$\begin{aligned} \hat{f}(x) &= f(x), && \text{if } x \in [a, b], \\ &= L_f(x), && \text{if } x \notin [a, b]. \end{aligned}$$

Since $L_f * \tilde{K}_n = L_f$, one can easily show that $\hat{f} * \tilde{K}_n = L_f + (\hat{f} - L_f) * \tilde{K}_n = L_f + \bar{f} * \tilde{K}_n$, and we shall denote the polynomial $\hat{f} * \tilde{K}_n$ on $[a, b]$ by f_n, and the polynomial f_n is called the nth *Landau polynomial* for f.

For any polynomials p and q such that $p(a) = f(a)$ and $q(b) = f(b)$, let $f(p; q)$ be the extension of f defined by

$$\begin{aligned} f(p; q)(x) &= p(x), && \text{if } x < a \\ &= f(x), && \text{if } x \in [a, b] \\ &= q(x), && \text{if } x > b. \end{aligned}$$

Let $f_n(p; q)$ denote the convolution $f(p; q) * \tilde{K}_n$. It is easy to see that $f_n(p; q)$ is a polynomial of degree $\leq 2n + 1 + \max \operatorname{degr}(p, q)$. Also, since $\{\tilde{K}_n\}$ is a *Dirac sequence* and since $\tilde{K}_n(x - t)$ as a function of t has support in $[2a - b, 2b - a]$ for each $x \in [a, b]$, $f_n(p; q) \to f$ uniformly on $[a, b]$ and

$$f_n(p; q)(x) = \int_{2a-b}^{2b-a} f(p; q)(t)\tilde{K}_n(x - t)\,dt.$$

We shall call $f_n(p; q)$ a redefined *Landau polynomial.* Note that when $p = q = L_f$, then $f_n(p; q)$ is the original nth *Landau polynomial* f_n. However, when f is a polynomial of degree i, f_n is usually of degree $2n$, but for each n the redefined Landau polynomial $f_n(f; f)$(with $p = q = f$) is of degree i and has the same ith and $(i - 1)$th derivatives as f.

2.

The following lemma is well known (see [1, p. 106] or [3, p. 54]).

LEMMA 1. *Let T be in $\mathscr{D}'(a, b)$.*

(i) *$\mathscr{D}T \geq 0$ in (a, b) if and only if T is defined by a nondecreasing function in (a, b).*

(ii) *$\mathscr{D}^2T \geq 0$ in (a, b) if and only if T is defined by a convex function in (a, b).*

Remark. Strict inequality (i.e., positive definiteness) in Lemma 1 characterizes the increasing functions and the strictly convex functions.

Let L be defined by $L(t) = (b - a)t + a$.

THEOREM 2. *Let f and g be in $C(a, b)$ and suppose that for some integer $m \geq 0$,*

$$\mathscr{D}^m T_f \geq T_g \qquad in\ (a, b).$$

(i) *If $m = 1$, then f is locally of bounded variation in (a, b).*

(ii) *If $m \geq 2$, then $f \in C^{m-2}(a, b)$ and $f^{(m-2)}$ is locally Lipschitzian on (a, b).*

(iii) *If m is any nonnegative integer, then*

$$\mathscr{D}^m T_{f\circ L} \geq (b - a)^m\, T_{g\circ L} \qquad in\ (0, 1).$$

Proof. (i) Let ζ be such that $\zeta' = g$ in (a, b). Then $\mathscr{D}T_{f-\zeta} \geq 0$ in (a, b), and by Lemma 1(i) we see that $f - \zeta$ is nondecreasing. The result now follows from the monotonicity of $f - \zeta$ since $\zeta \in C^1(a, b)$.

(ii) Let H be any solution to the differential equation $y^{(m)} = g$ in (a, b). Then $\mathscr{D}^m T_{f-H} \geq 0$ since $\mathscr{D}^m T_H = T_g$. By Lemma 1(ii), we can conclude that the distribution $\mathscr{D}^{m-2}T_{f-H}$ is defined by a convex function κ on (a, b). Then the continuity of f implies that $(f - H)^{(m-2)} = \kappa$. Since convex functions are locally Lipschitzian and $H^{(m)}$ is continuous, we obtain that $f^{(m-2)}$ exists and is locally Lipschitzian.

(iii) $\mathscr{D}^m T_f \geq T_g$ in (a, b) implies that for any nonnegative test function ϕ on $(0, 1)$,

$$\begin{aligned}
\mathscr{D}^m T_{f\circ L}(\phi) &= (-1)^m \int_0^1 f((b-a)t + a)\, \phi^{(m)}(t)\, dt \\
&= \frac{(-1)^m}{(b-a)} \int_a^b f(u)\, \phi^{(m)}\left(\frac{u-a}{b-a}\right) du \\
&= (b-a)^{m-1}\, \mathscr{D}^m T_f\left(\phi\left(\frac{u-a}{b-a}\right)\right) \\
&\geq (b-a)^{m-1}\, T_g\left(\phi\left(\frac{u-a}{b-a}\right)\right),
\end{aligned}$$

since the function defined by $u \to \phi((u - a)/(b - a)) \geq 0$ is in $\mathscr{D}(0, 1)$. Then because $T_g(\phi((u - a)/(b - a))) = (b - a)\, T_{g \circ L}(\phi)$, we have that $\mathscr{D}^m T_{f \circ L} \geq (b - a)^m\, T_{g \circ L}$ in $(0, 1)$. The proof is complete.

THEOREM 3. *Let f be in $C[a, b]$ and suppose that for some integer $m \geq 0$, there exists $g \in C[a, b]$ such that*

$$\mathscr{D}^m T_{\tilde{f}} \geq T_{\tilde{g}} \qquad \text{in } (2a - b, 2b - a).$$

Then for each integer $n > m$, $f_n^{(m)} \geq g_n$ holds in (a, b), where $\{f_n\}$ and $\{g_n\}$ are the Landau polynomials *for f and g, respectively, on $[a, b]$. Furthermore, $f' \in L^1_{\text{loc}}(a, b)$ when $m = 1$ and when $m \geq 2$, $f \in C^{m-2}(a, b)$ and $f_n^{(i)} \to f^{(i)}$ uniformly on compact subintervals of (a, b) for each i $(1 \leq i \leq m - 2)$.*

Proof. First assume that $[a, b] = [0, 1]$ and let $K_{n,\epsilon}(x - t)$ be the regularizations (as defined in [1, p. 56]) of $K_n(x - t)$ for fixed $x \in (0, 1)$. For ϵ sufficiently small $K_{n,\epsilon}(x - t)$ is a nonnegative test function in $\mathscr{D}(-1, 2)$ as a function of t for $x \in (0, 1)$ because $K_n(x - t) \geq 0$ and has support $= [x - 1, x + 1] \subset (-1, 2)$. Furthermore, for $n > m$ and fixed $x \in (0, 1)$ we have that $K_{n,\epsilon}^{(m)}(x - t) \to K_n^{(m)}(x - t)$ uniformly on $[-1, 2]$ as $\epsilon \to 0$. Now we can conclude that for $n > m$,

$$\begin{aligned}
f_n^{(m)}(x) &= \tilde{f} * K_n^{(m)}(x) = \int_{-1}^{2} \tilde{f}(t)\, K_n^{(m)}(x - t)\, dt \\
&= (-1)^m \int_{-1}^{2} \tilde{f}(t)\, D_t^{(m)}(K_n(x - t))\, dt \\
&= \lim_{\epsilon \to 0} (-1)^m \int_{-1}^{2} \tilde{f}(t)\, D_t^{(m)}(K_{n,\epsilon}(x - t))\, dt \\
&= \lim_{\epsilon \to 0} \mathscr{D}^m T_{\tilde{f}}(K_{n,\epsilon}(x - t)) \geq \lim_{\epsilon \to 0} T_{\tilde{g}}(K_{n,\epsilon}(x - t)) \\
&= \tilde{g} * K_n(x) = g_n(x) \text{ for } x \in (0, 1).
\end{aligned}$$

The differentiability properties follow from Theorem 2 and permit us to conclude that $f_n^{(i)} = \tilde{f}^{(i)} * K_n \to f^{(i)}$ uniformly on compact subintervals of $(0, 1)$ because $f^{(i)}$ is in $C(0, 1)$(see [2, p. 212]).

In the general case when $[a, b]$ is any compact interval, we have from parts (ii) and (iii) of Theorem 2 that $f \in C^{m-2}(a, b)$ (when $m \geq 2$), and

$$\mathscr{D}^m T_{\tilde{f} \circ L} \geq (b - a)^m\, T_{\tilde{g} \circ L} \qquad \text{in } (-1, 2).$$

Let

$$L^{-1}(x) = \frac{x - a}{b - a}\,;$$

then $(f \circ L)_n(L^{-1}(x)) = f_n(x) = (\hat{f} * \tilde{K}_n)(x)$ for $x \in [a, b]$, and it follows from the part of the theorem already proved that

$$f_n^{(m)}(x) = (f \circ L)_n^{(m)}(L^{-1}(x)) \frac{1}{(b-a)^m} = (g \circ L)_n(L^{-1}(x))$$
$$= g_n(x) = \hat{g} * \tilde{K}_n(x).$$

Since $f_n^{(i)} = \hat{f}^{(i)} * \tilde{K}_n$, we also obtain from the case $[a, b] = [0, 1]$ that $f_n \to f$ uniformly on $[a, b]$ and $f_n^{(i)} \to f^{(i)}$ uniformly on compact subintervals of (a, b) for each i $(1 \leq i \leq m - 2)$.

Remark. Note that $\hat{f}$ is monotone on $(2a - b, 2b - a)$ whenever f is monotone on (a, b). So Lemma 1 and Theorem 3 imply that each of the original Landau polynomials f_n is monotone on $[a, b]$ when f is.

THEOREM 4. *Let f be in $C[a, b]$ and suppose that for some integer $m \geq 0$, there exists $g \in C[a, b]$ such that*

$$\mathcal{D}^m T_f \geq T_g \qquad \textit{in } (a, b).$$

Then there exists a sequence of redefined Landau polynomials $\{\tilde{f}_n\}$ depending on m and g such that $\tilde{f}_n^{(m)} \geq \tilde{g}_n$ holds in (a, b) for each n, where $\{\tilde{g}_n\}$ is a redefined sequence of Landau polynomials which converges uniformly to g on $[a, b]$. Furthermore, $f' \in L^1_{\text{loc}}(a, b)$ when $m = 1$ and when $m \geq 2$, $f \in C^{m-2}(a, b)$ and $\tilde{f}_n^{(i)} \to f^{(i)}$ uniformly on compact subintervals of (a, b) for each $i(0 \leq i \leq m - 2)$.

Proof. The differentiability properties of f follow from Theorem 2. From the proof of (ii) of that theorem we know that $\kappa = (f - H)^{(m-2)}$ is convex on (a, b). Since κ is convex, it is differentiable a.e., and the left- and right-hand derivatives exist everywhere in (a, b) and are nondecreasing. Thus κ'' is defined a.e. in (a, b). This implies that $f^{(m)}$ exists a.e. in (a, b) because $H \in C^m(a, b)$.

Let $\{(a_k, b_k)\}$ be a sequence of intervals which increase to (a, b) and such that $f^{(m)}$ exists at each a_k and b_k. For each k, define $\bar{f}(p_k ; q_k)$ by

$$\begin{aligned} \bar{f}(p_k ; q_k)(x) &= p_k(x), && \text{if } x < a_k, \\ &= f(x), && \text{if } x \in [a_k, b_k], \\ &= q_k(x), && \text{if } x > b_k, \end{aligned}$$

where p_k and q_k are the mth Taylor polynomials for f at a_k and b_k, respectively. Let g_k be $g(a_k)$ for $x < a_k$, $g(b_k)$ for $x > b_k$, and agree with g on $[a_k, b_k]$. If H_k is a C^m function such that $H_k^{(m)} = g_k$, then $\kappa_k = (\bar{f}(p_k ; q_k) - H_k)^{(m-2)}$ is convex on $(-\infty, +\infty)$ because $\kappa_k'' \geq 0$ outside of $[a_k, b_k]$, and because κ_k equals $(f - H)^{(m-2)}$ on (a_k, b_k) and is differentiable at the endpoints. Now it follows from Lemma 1 and $(m - 2)$ integrations by parts that the mth distributional derivative of $\bar{f}(p_k\,;q_k)$ is greater than or equal to (in the sense of distributions) the distribution defined by g_k for each integer $k \geqslant 0$. From this we obtain, just as in the proof of Theorem 3, that $(\bar{f}(p_k; q_k) * \tilde{K}_n)^{(m)} \geqslant g_k * \tilde{K}_n$ holds in (a, b) for any $n \geqslant 0$. Hence the sequence defined by $\tilde{f}_n = \bar{f}(p_n\,; q_n) * \tilde{K}_n$ satisfies $\tilde{f}_n^{(m)} \geqslant \tilde{g}_n$ in (a, b) and has the desired convergence properties since $\bar{f}(p_n; q_n) = f$ (for n sufficiently large) on each compact subinterval of (a, b) and $\tilde{g}_n = g_n * \tilde{K}_n \to g$ uniformly on $[a, b]$. This completes the proof.

Remark. Lemma 1 and Theorems 2, 3, and 4 remain valid when we reverse the inequalities and replace the words "nondecreasing" and "convex" by "nonincreasing" and "concave" wherever they appear.

Theorem 3 shows that the original Landau polynomials inherit their properties from $\bar{f}$, so, for example, they might not be convex on $[a, b]$ when f is convex on $[a, b]$. Theorem 4 shows how to approximate by redefined Landau polynomials having one prescribed property on $[a, b]$ possessed by f. The following theorem treats (simultaneously) special prescribed properties.

THEOREM 5. *Let f be in $C[a, b]$. If f has any of the properties*

(i) *odd or even where $b = -a > 0$,*

(ii) *nondecreasing or nonincreasing,*

(iii) *convex or concave*

on $[a, b]$, then there exists a sequence of redefined Landau polynomials $\{\tilde{f}_n\} \to f$ uniformly on $[a, b]$ such that each $\tilde{f}_n$ has the corresponding property on $[a, b]$.

Proof. For $2/k \leqslant b - a$, let p_k be the function determined by the line through the points $(a, f(a))$ and $(a_k\,, f(a_k))$, and let q_k be determined by the line through $(b, f(b))$ and $(b_k\,, f(b_k))$, where $a_k = a + 1/k$ and $b_k = b - 1/k$. Then it is easy to see that each $\bar{f}(p_k\,; q_k)$ (as defined in Theorem 4) inherits on $[2a - b, 2b - a]$ any of the properties (i), (ii), and (iii) from f. (By renumbering we may assume that $\bar{f}(p_k\,; q_k)$ is defined for each positive integer k.) Now it follows, from Lemma 1, Theorem 4, and obvious facts about the convolutions of even or odd functions with even functions, that $\bar{f}(p_k\,; q_k) * \tilde{K}_n$ inherits on $[a, b]$ any of these properties from f for any positive integers n and k. Since $\bar{f}(p_k\,; q_k) \to f$ uniformly on $[a, b]$ and $\{\tilde{K}_n\}$ is a *Dirac sequence*, we can select from $\{\bar{f}(p_k\,; q_k) * \tilde{K}_n\}$ a sequence $\{\tilde{f}_n\}$ which converges uniformly to f on $[a, b]$.

Remarks. (1) It is obvious that if each of the polynomials satisfies the hypotheses of either Theorem 3 or Theorem 4 or has any of the properties in Theorem 5, then f does also.

(2) The results given here easily generalize

(i) when $f \in L^1[a, b]$ and convergence is in L^1 norm;

(ii) in higher dimensions, when the functions are continuous on a finite product of compact intervals (see [4, p. 123]).

(3) Since the function $-x^{1/2}$ is convex on [0, 1] but cannot be extended to a convex function on $[a, 1]$ for any $a < 0$, we see why the convolutions $\bar{f}(p_k; q_k) * \tilde{K}_n$ were used in the proof of Theorem 5.

(4) The degree of approximation of continuous functions by certain convolution-type operators (including the Landau kernel) can be found in [6].

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